

Equality statements for entropy change in open systems

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Abstract

The entropy change of a (non-equilibrium) Markovian ensemble is calculated from (1) the ensemble phase density $p(t)$ evolved as iterative map, $p(t) = \mathbb{M}(t)p(t-\Delta t)$ under detail balanced transition matrix $\mathbb{M}(t)$, and (2) the invariant phase density $\pi(t) = \mathbb{M}(t)^\infty \pi(t)$. A virtual measurement protocol is employed, where variational entropy is zero, generating exact expressions for irreversible entropy change in terms of the Jeffreys measure, $\mathcal{J}(t) = \sum_\Gamma [p(t) - \pi(t)] \ln [p(t)/\pi(t)]$, and for reversible entropy change in terms of the Kullback-Leibler measure, $\mathcal{D}_{KL}(t) = \sum_\Gamma \pi(0) \ln [\pi(0)/\pi(t)]$. Five properties of \mathcal{J} are discussed, and Clausius' theorem is derived.

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Reversible manipulation is the principal tool of the thermodynamicist. Reversibility appears in two forms: the quasi-static time-forward reversible transition and the microscopically-reversible time-reversed (or adjoint) stochastic transition [1, 2, 3]. Application of microscopic reversibility to the path integral formulation of stochastic processes [4, 5] has resulted in a set of fluctuation theorems (FT) for systems arbitrarily far from equilibrium [6]. FT, despite their elegance, do not provide a much needed *general definition of entropy change*—an equality statement providing the entropy change for any transition—of an ensemble of Markovian systems. Here, using both the quasi-static and time-reversed transitions in the path integral approach to the dynamics of a Markovian system, we produce these equality expressions for microscopic and macroscopic entropy change.

A collection of M classical particles undergoing Hamiltonian dynamics is partitioned, through scale separation, into system and bath [7, 8]. The system, consisting of N particles is transformed into a Markovian stochastic process described by $6N$ generalized coordinates. Phase space and time are taken as discrete quantities. Each coordinate is an m -tuple, and time consists of equally spaced intervals, $\Delta\tau = \tau_{i+1} - \tau_i$. The system trajectory is given as the time evolution of a phase point, $\sigma(\tau) = \delta[(x; \tau) - (x_0; \tau)]$, in phase space Γ , a $(6N \times m)$ -tuple, with $x, x_0 \in \Gamma$, according to the stochastic iterative map

$$\sigma(\tau_i) = \mathbb{M}_{\tau_i} \sigma(\tau_{i-1}), \quad (1)$$

where $\mathbb{M}_{\tau_i} = \mathbb{M}_{\tau_i}(\sigma(\tau_i) | \sigma(\tau_{i-1}))$ is interpreted as a stochastic matrix [9]. Real systems, which operate under colored (OU) noise, are managed by requiring that the discrete time step in (1) be much longer than the correlation time of the noise. Following Gibbs, we consider an ensemble of such collections. Interpreting \mathbb{M}_{τ_i} as a transition matrix (rather than a stochastic matrix) [10] and defining the phase probability as the normalized density of phase points, $P(\tau_i) = \overline{\sigma(\tau_i)}$, the dynamics of the ensemble is a time-inhomogeneous Markov chain

$$P(\tau_i) = \mathbb{M}_{\tau_i} P(\tau_{i-1}). \quad (2)$$

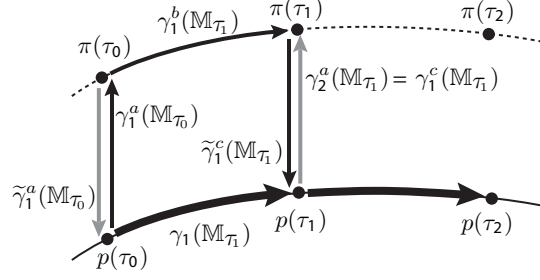


FIG. 1: Ensemble undergoing a general perturbation. Evolution of phase density p along path γ , consisting of small displacements γ_i . The entropy change under γ_i is measured along the virtual paths γ_i^a , γ_i^b , $\tilde{\gamma}_i^c$ involving the invariant density π .

It is assumed that \mathbb{M}_{τ_i} is a known quantity obtained through experimental parametrization or on the basis of theory. From \mathbb{M}_{τ_i} and given starting phase density $p(\tau_0) = P(\tau_0)$, two time-dependent quantities of interest are determined—the time-dependent phase density $p(\tau_i)$ and the time-dependent invariant phase density, $\pi(\tau_i) = \mathbb{M}_{\tau_i}^\infty \pi(\tau_i)$, obtained as follows: the dynamics at time τ_i are stopped, then the density is evolved in virtual time $t_i \rightarrow \infty$ under stationary \mathbb{M}_{τ_i} , according to (2). For nonequilibrium system ensembles, π is a virtual quantity. For equilibrium ensembles that undergo quasi-static perturbation, π is a real quantity. Our results apply to ensembles that evolve according to (2) with transition matrices \mathbb{M}_{τ_i} that are Hermitian. These systems possess three important properties [11]. (i) Microscopic reversibility— \mathbb{M}_{τ_i} is self-adjoint [3, 4]: $\mathbb{M}_{\tau_i} \pi(\tau_i) = \tilde{\mathbb{M}}_{\tau_i} \pi(\tau_i)$. (ii) Invariance—the invariant distribution of a stationary Markov process is independent of the ensemble history: $\lim_{t_n \rightarrow \infty} \mathbb{M}_{\tau_i}^{t_n} p(\tau_i) = \pi(\tau_i)$. (iii) Stationarity—with invariant density: $\pi(\tau_i) = \mathbb{M}_{\tau_i} \pi(\tau_i)$. Microscopic reversibility is a property of physical systems [12] and a fundamental postulate of physics [2].

General Perturbation—With these properties in mind, we calculate the entropy change of a thermodynamic ensemble of Markovian systems undergoing an arbitrary forced perturbation. Using the prescription in Fig. 1, entropy change for the ensemble transition along the path increment γ_i (bold arrows) is evaluated using three measurements that,

being virtual, do not perturb the system [1, 2]. The path γ taken by one system starting at $\sigma(t_0)$ is given by the time-ordered collection of phase points visited by the system $\{\sigma(t_0)\sigma(t_1)\cdots\sigma(t_n)\}$ under evolution by $\mathbb{M}_{t_0}\mathbb{M}_{t_1}\cdots\mathbb{M}_{t_n}$,

$$\gamma \equiv \sigma(t_0) \xrightarrow{\mathbb{M}_{t_1}} \sigma(t_1) \xrightarrow{\mathbb{M}_{t_2}} \cdots \sigma(t_{n-1}) \xrightarrow{\mathbb{M}_{t_n}} \sigma(t_n).$$

The adjoint path is

$$\tilde{\gamma} \equiv \tilde{\sigma}(t_n) \xleftarrow{\tilde{\mathbb{M}}_{t_n}} \tilde{\sigma}(t_{n-1}) \cdots \xleftarrow{\tilde{\mathbb{M}}_{t_2}} \tilde{\sigma}(t_1) \xleftarrow{\tilde{\mathbb{M}}_{t_1}} \tilde{\sigma}(t_0),$$

where $\tilde{\mathbb{M}}_{t_j} = \tilde{\mathbb{M}}_{t_j}(\tilde{\sigma}(t_j)|\tilde{\sigma}(t_{j-1}))$. The adjoint path starts where γ ends, $\tilde{\sigma}(t_0) = \sigma(t_n)$. For a system in state $\sigma(t_0)$, the probability that it follows the path γ is given by the product of the single time step transition probabilities, $\mathcal{P}(\gamma|p) = \prod_{i=1}^n \mathbb{M}_{t_i}$, where $p = P(\sigma(t_0))$. Similarly, for the adjoint transition, the conditional adjoint path probability is $\mathcal{P}(\tilde{\gamma}|\tilde{p}) = \prod_{i=1}^n \tilde{\mathbb{M}}_{t_i}$, where $\tilde{p} = P(\tilde{\sigma}(t_0))$ is the probability that the system starts in state $\tilde{\sigma}(t_0)$. Using the definition of conditional probability and taking the quotient of path probabilities, we obtain

$$\frac{\mathcal{P}(\gamma)}{\mathcal{P}(\tilde{\gamma})} = \frac{\mathcal{P}(\gamma|p)p}{\mathcal{P}(\tilde{\gamma}|\tilde{p})\tilde{p}} = \prod_{i=1}^n \frac{\mathbb{M}_{t_i} p}{\tilde{\mathbb{M}}_{t_i} \tilde{p}}. \quad (3)$$

The conditional path probability is also given as a function of action [1, 4], $\mathcal{P}(\gamma|p) = \exp(-\sum_i \mathcal{A}_{t_i}(\gamma))$, and we note the correspondence, $\ln \mathbb{M}_{t_i} = -A_{t_i}$. Defining the microscopic entropy change of the collection [4, 5, 6, 16]

$$\begin{aligned} \tilde{\delta}\mathcal{S} &\equiv \ln \mathcal{P}(\gamma) - \ln \mathcal{P}(\tilde{\gamma}) \\ &= \ln \left[\frac{p}{\tilde{p}} e^{\sum_i \ln \mathbb{M}_{t_i} - \ln \tilde{\mathbb{M}}_{t_i}} \right], \end{aligned} \quad (4)$$

we obtain a microscopic entropy balance equation

$$\tilde{\delta}\mathcal{S} = \Delta\mathcal{S}_\gamma + \ln [p/\tilde{p}], \quad (5)$$

involving gain of entropy by the heat bath, $\Delta\mathcal{S}_\gamma = \sum_i \ln \mathbb{M}_{t_i}/\tilde{\mathbb{M}}_{t_i}$, and gain of entropy by the system, $\ln(p/\tilde{p})$. In (4), entropy and action are on equal footing: entropy is proportional to the logarithm of exponentiated action. Evaluation of (5) is straightforward when some

path γ can be identified where $\tilde{\delta}\mathcal{S} = 0$. We show that the measurement in Fig. 1 is along such a path.

Each path γ_i^a involves the evolution in virtual time t_0, t_1, \dots, t_n of the ensemble starting in $p(\tau_{i-1})$ to the stationary distribution $\pi(\tau_{i-1})$. The evolution is a virtual time-homogeneous transition governed by the transition matrix $\mathbb{M}_{\tau_{i-1}}$. For each γ_i^a , (5) provides, $\Delta\mathcal{S}_{\gamma_i^a} = \tilde{\delta}\mathcal{S}_{\gamma_i^a} - \ln[p(\tau_{i-1})/\pi(\tau_{i-1})]$. Using the property of invariance (ii), the path probability density is

$$\mathcal{P}(\gamma_i^a) = \mathbb{M}_{\tau_{i-1}}^\infty p(\tau_{i-1}) = \pi(\tau_{i-1}). \quad (6)$$

Application of detailed balance (i) followed by sequential application of stationarity (iii) to the adjoint path probability density yields

$$\begin{aligned} \mathcal{P}(\tilde{\gamma}_i^a) &= \tilde{\mathbb{M}}_{\tau_{i-1}}^\infty \pi(\tau_{i-1}) = \mathbb{M}_{\tau_{i-1}}^\infty \pi(\tau_{i-1}) \\ &= \pi(\tau_{i-1}). \end{aligned} \quad (7)$$

For each path γ_i^a , the microscopic entropy change of the collection is zero, $\tilde{\delta}\mathcal{S}_{\gamma_i^a} = \ln[\mathcal{P}(\gamma_i^a)/\mathcal{P}(\tilde{\gamma}_i^a)] = 0$, yielding, $\Delta\mathcal{S}_{\gamma_i^a} = -\ln[p(\tau_{i-1})/\pi(\tau_{i-1})]$. The macroscopic entropy over the disjoint paths γ_i^a is the ensemble averaged entropy

$$\Delta\mathcal{S}_{\gamma^a} = -\sum_{i=0}^{n-1} \langle \ln[p(\tau_i)/\pi(\tau_i)] \rangle_{p(\tau_i)}, \quad (8)$$

where $\langle f(\Gamma) \rangle_{x(\Gamma)} = \sum_{\Gamma} x(\Gamma) f(\Gamma)$.

Each path γ_i^b involves the virtual evolution of the invariant starting distribution $\pi(\tau_{i-1})$ to the invariant distribution $\pi(\tau_i)$ under (virtual) time-homogeneous evolution by \mathbb{M}_{τ_i} . For each γ_i^b , (5) provides, $\Delta\mathcal{S}_{\gamma_i^b} = \tilde{\delta}\mathcal{S}_{\gamma_i^b} - \ln[\pi(\tau_{i-1})/\pi(\tau_i)]$. By the same arguments used in (6) and (7), for each path γ_i^b , $\tilde{\delta}\mathcal{S}_{\gamma_i^b} = \ln[\mathcal{P}(\gamma_i^b)/\mathcal{P}(\tilde{\gamma}_i^b)] = 0$, yielding, $\Delta\mathcal{S}_{\gamma_i^b} = -\ln[\pi(\tau_{i-1})/\pi(\tau_i)]$. We concatenate the γ_i^b path segments into a continuous virtual path γ^b for the evolution of π . The microscopic entropy over the thermodynamically reversible path γ^b is, after cancelling terms, $\Delta\mathcal{S}_{\gamma^b} = \sum_{i=1}^n \Delta\mathcal{S}_{\gamma_i^b} = -\ln[\pi(\tau_0)/\pi(\tau_n)]$. The

reversible macroscopic entropy flow *into the system* during γ^b is

$$\Delta \mathbf{S}_{rev} = -\Delta \mathbf{S}_{\gamma^b} = \langle \ln [\pi(\tau_0)/\pi(\tau_n)] \rangle_{\pi(\tau_0)}, \quad (9)$$

which is the relative, or Kullbach-Leibler, entropy $\mathcal{D}_{KL}[\mathbf{x}, \mathbf{y}] = \sum_{\Gamma} \mathbf{x} \ln(\mathbf{x}/\mathbf{y})$ [17].

Each path γ_i^c involves the adjoint (virtual) time-homogeneous evolution of the ensemble starting from the invariant distribution $\pi(\tau_i)$ to the real distribution $p(\tau_i)$ under $\tilde{\mathbb{M}}_{\tau_i}$. For each γ_i^c the entropy (5) is, $\Delta \mathbf{S}_{\gamma_i^c} = \tilde{\delta} \mathbf{S}_{\gamma_i^c} - \ln [\pi(\tau_i)/p(\tau_i)]$. Again, by the same arguments used in (6) and (7), for each path γ_i^c , $\tilde{\delta} \mathbf{S}_{\gamma_i^c} = \ln [\mathcal{P}(\gamma_i^c)/\mathcal{P}(\tilde{\gamma}_i^c)] = 0$, yielding, $\Delta \mathbf{S}_{\gamma_i^c} = -\ln [\pi(\tau_i)/p(\tau_i)]$. The macroscopic entropy change over the disjoint paths γ_i^c is

$$\Delta \mathbf{S}_{\gamma^c} = -\sum_{i=1}^n \langle \ln [\pi(\tau_i)/p(\tau_i)] \rangle_{\pi(\tau_i)}. \quad (10)$$

From (8) and (10), the irreversible macroscopic entropy flow *into the system* over γ^a and γ^c , $\Delta \mathbf{S}_{irrev} = -(\Delta \mathbf{S}_{\gamma^a} + \Delta \mathbf{S}_{\gamma^c})$, is

$$\Delta \mathbf{S}_{irrev} = B_0 + \sum_{i=1}^{n-1} \langle \ln [p(\tau_i)/\pi(\tau_i)] \rangle_{p(\tau_i)-\pi(\tau_i)} + B_n, \quad (11)$$

where $B_0 = \sum_{\Gamma} p(\tau_0) \ln p(\tau_0)/\pi(\tau_0)$ and $B_n = \sum_{\Gamma} \pi(\tau_n) \ln \pi(\tau_n)/p(\tau_n)$ are boundary Kullbach-Leibler integrals. The sum in (11) is over the Jeffreys invariant divergence measure, $\mathcal{J}[\mathbf{x}, \mathbf{y}] = \sum_{\Gamma} (\mathbf{x} - \mathbf{y}) \ln(\mathbf{x}/\mathbf{y})$ [18]. Jeffreys [19] and others [17] have commented on the many remarkable properties of \mathcal{J} .

Perturbation #1—We apply the results obtained for the general perturbation to two specific, and important, perturbations (Fig. 2). In Fig. 2a, the system is perturbed from one equilibrium state to another. For the (virtual) equilibrium path (dashed line), macroscopic entropy flow into the system is, from (9), the Kullbach-Leibler entropy, $\Delta \mathbf{S}_{rev} = \mathcal{D}_{KL}[\pi(\tau_0), \pi(\tau_n)]$. For the (real) non-equilibrium path (solid line), macroscopic entropy is the sum of the reversible entropy (9) and the irreversible entropy (11), $\Delta \mathbf{S}_{tot} = \Delta \mathbf{S}_{rev} + \Delta \mathbf{S}_{irrev}$. Here, $p(\tau_0) = \pi(\tau_0)$ and $p(\tau_n) = \pi(\tau_n)$, causing B_0 and B_n to vanish. The irreversible entropy flow into the system is the discrete time integral over the

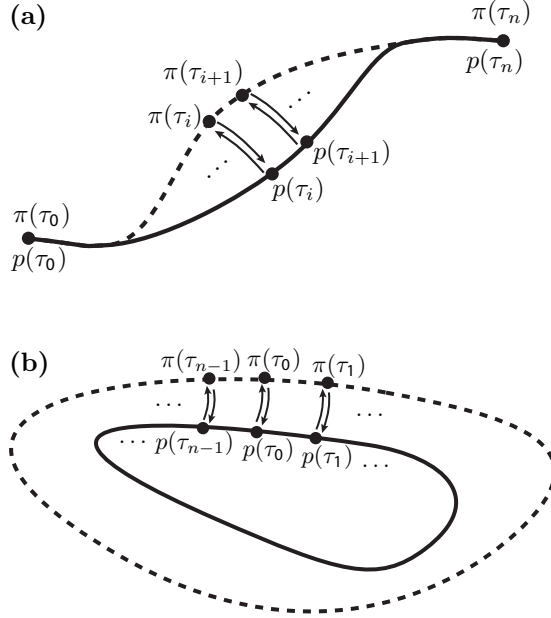


FIG. 2: Two perturbations. (a) Non-equilibrium perturbation with p starting and ending in equilibrium (convergence of p and π). (b) Periodic perturbation with closed orbits of p and π .

Jeffreys invariant measure, $\Delta \mathbf{S}_{irrev} = \sum_{i=1}^{n-1} \mathcal{J}[p(\tau_i), \pi(\tau_i)]$. The total entropy flow into the system is

$$\Delta \mathbf{S}_{tot} = \mathcal{D}_{KL}[\pi(\tau_0), \pi(\tau_n)] + \sum_{i=1}^{n-1} \mathcal{J}[p(\tau_i), \pi(\tau_i)]. \quad (12)$$

Perturbation #2—In Fig. 2b, we consider an ensemble of systems undergoing periodic perturbation with period $\omega \Delta \tau$, where $\mathbb{M}_{\tau_i} = \mathbb{M}_{\tau_i + \omega}$. Using (2), an initial phase density $p(\tau_0) = P(\sigma(\tau_0))$, is prepared from some arbitrary phase density $p(\tau_{-\Omega})$ through the equilibration process: $p(\tau_0) = (\mathbb{M}_{\tau_\omega} \mathbb{M}_{\tau_{\omega-1}} \dots \mathbb{M}_{\tau_1})^{\Omega/\omega} p(\tau_{-\Omega})$, where $\Omega/\omega \in \mathbb{I}^+ \gg 1$. While the Poincaré recurrence time for any one system may be extremely long, the recurrence time for the ensemble is $\omega \Delta \tau$. From (9) and using the property, $\pi(\tau_0) = \pi(\tau_n)$, we obtain $\Delta \mathbf{S}_{rev} = \mathcal{D}_{KL}[\pi(\tau_0), \pi(\tau_n)] = 0$. From (11) and the property, $p(\tau_0) = p(\tau_n)$, we obtain $\Delta \mathbf{S}_{irrev} = \sum_{i=1}^n \mathcal{J}[p(\tau_i), \pi(\tau_i)]$. The total entropy flow into the system over one cycle of

perturbation is

$$\Delta \mathbf{S}_{tot} = \sum_{i=1}^n \mathcal{J}[p(\tau_i), \pi(\tau_i)]. \quad (13)$$

The total entropy transferred from the bath to the system over a thermodynamic cycle is the time integral of the Jeffreys divergence between real and invariant phase densities.

Properties of \mathcal{J} —The properties of \mathcal{J} generate some important conclusions. \mathcal{J} is almost positive definite, meaning (i) $\mathcal{J}[\mathbf{x}, \mathbf{y}] \geq 0$ and (ii) $\mathcal{J}[\mathbf{x}, \mathbf{y}] = 0$ only when $x = y$. (iii) \mathcal{J} is symmetric: $\mathcal{J}[\mathbf{x}, \mathbf{y}] = \mathcal{J}[\mathbf{y}, \mathbf{x}]$. (iv) \mathcal{J} is a linear measure: $\det(\partial^2 \mathcal{J} / \partial \mathbf{x}_i \partial \mathbf{x}_j) = 0$. \mathcal{D}_{KL} satisfies (i), (ii) [20] and (iv). See EPAPS Document No. [] for proofs and further discussion.

For an isothermal (canonical) system, $\Delta \mathbf{S} = -\Delta \mathbf{S}_{tot}$, is the heat flow per temperature from the system to the bath during a periodic perturbation, $\Delta \mathbf{S} = \oint \beta d\mathbf{Q}$, where $\beta = 1/k_B T$ is inverse temperature in units of energy. Clausius' statement of the second law of thermodynamics is obtained from (13) and properties (i) and (ii),

$$\oint \beta d\mathbf{Q} \leq 0.$$

Practical application of (13) to many-body systems derives from a fifth property of \mathcal{J} , (v) decomposability [19]: for a system with a decomposable Markov transition matrix

$$\mathbb{M} = \begin{bmatrix} \mathbb{A} & \mathbf{0} \\ \mathbf{0} & \mathbb{B} \end{bmatrix},$$

the phase density decomposes: $p = p_A p_B$, $\pi = \pi_A \pi_B$, phase space decomposes: $\Gamma = \Gamma_A \Gamma_B$, and the invariant measure decomposes: $\mathcal{J}[p, \pi] = \mathcal{J}^A[p_A, \pi_A] + \mathcal{J}^B[p_B, \pi_B]$. Defining, $\Delta \mathbf{S}^k = \sum_i \mathcal{J}_i^k[p_i, \pi_i]$, and using (v), we find that macroscopic entropy is extensive, $\Delta \mathbf{S} = \Delta \mathbf{S}^A + \Delta \mathbf{S}^B$. The most immediate application of decomposability is the overdamped system where Γ_B comprises momentum space and momentum is always equilibrated, $p_B = \pi_B$. Using (13) and properties (ii) and (v), we obtain, $\Delta \mathbf{S} = \Delta \mathbf{S}_A$. Further reduction may be possible with a suitable choice of basis for Γ_A [21] and also upon coarse graining of the system [7, 22].

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A. Some comments

Aside from generating equality statements for reversible and irreversible entropy change, perhaps our most important contribution is providing a purely virtual integration protocol for evaluating a variation. Being virtual, the integration can not introduce uncertainty into the system [1, 2]. This measurement protocol is likely to find application beyond statistical mechanics.

We note that Gibbs entropy, $\mathcal{S} = -\ln p$, [3, 4] and Boltzmann entropy, $\mathcal{S} = -\ln \pi$, follow as boundary terms from the definition of variational entropy $\tilde{\delta}\mathcal{S}$ when p or \tilde{p} are evaluated at π .

B. Some properties of the Jeffreys divergence \mathcal{J} and the Kullback-Leibler divergence \mathcal{D}

The Jeffreys divergence measure is defined

$$\mathcal{J}[\mathbf{x}, \mathbf{y}] = \sum_{\Gamma} (\mathbf{x} - \mathbf{y}) \ln(\mathbf{x}/\mathbf{y}).$$

The Kullback-Leibler divergence measure is defined

$$\mathcal{D}[\mathbf{x}, \mathbf{y}] = \sum_{\Gamma} \mathbf{x} \ln(\mathbf{x}/\mathbf{y}).$$

Theorem .1 \mathcal{J} is almost positive definite, meaning (i) $\mathcal{J}[\mathbf{x}, \mathbf{y}] \geq 0$ and (ii) $\mathcal{J}[\mathbf{x}, \mathbf{y}] = 0$ only when $\mathbf{x} = \mathbf{y}$.

Proof We consider, element-wise, the probability densities $\mathbf{x} = \{x\}$, $\mathbf{y} = \{y\}$, and the Jeffreys measure, $\mathcal{J}[\mathbf{x}, \mathbf{y}] = \sum_{\Gamma} \mathcal{J}_i[x, y]$. For $x > y$, $(x - y) > 0$ and $\ln(x/y) > 0$; therefore, $\mathcal{J}_i[x, y] = (x - y) \ln(x/y) > 0$. For $x < y$, $(x - y) < 0$ and $\ln(x/y) < 0$; therefore, $\mathcal{J}_i[x, y] > 0$. For $x = y$, $(x - y) = 0$ and $\ln(x/y) = 0$; therefore, $\mathcal{J}_i[x, y] = 0$. ■

Theorem .2 \mathcal{J} is symmetric: $\mathcal{J}[\mathbf{x}, \mathbf{y}] = \mathcal{J}[\mathbf{y}, \mathbf{x}]$

Proof \mathcal{J} is evaluated element-wise. Both terms $(x-y)$ and $\ln(x/y)$ are odd under exchange of x and y . The product of two odd functions is even. ■

Theorem .3 \mathcal{J} does not satisfy the triangle inequality: $\mathcal{J}[x, z] \leq \mathcal{J}[x, y] + \mathcal{J}[y, z]$.

Proof The proof is by example (Nikolai Chernov, personal communication). Let $x = [0.25, 0.75]$, $y = [0.50, 0.50]$, and $z = [0.75, 0.25]$. $\mathcal{J}[x, y] = 0.27$; $\mathcal{J}[y, z] = 0.27$; $\mathcal{J}[x, z] = 1.10$. We obtain, $\mathcal{J}[x, z] > \mathcal{J}[x, y] + \mathcal{J}[y, z]$. ■

By not satisfying the triangle equality, the Jeffreys measure falls short of being a topologic metric [5]. For this reason, the term “Jeffreys divergence measure” is the preferred over the “Jeffreys distance measure.”

Theorem .4 \mathcal{J} is a linear measure.

Proof The Hession of $\mathcal{J}[x, y]$,

$$H(\mathcal{J}[x, y]) = \begin{bmatrix} \partial^2 \mathcal{J} / \partial x \partial x & \partial^2 \mathcal{J} / \partial x \partial y \\ \partial^2 \mathcal{J} / \partial y \partial x & \partial^2 \mathcal{J} / \partial y \partial y \end{bmatrix},$$

is evaluated: $\partial^2 \mathcal{J} / \partial x \partial x = (x + y)/x^2$, $\partial^2 \mathcal{J} / \partial y \partial y = (x + y)/y^2$, and $\partial^2 \mathcal{J} / \partial x \partial y = \partial^2 \mathcal{J} / \partial y \partial x = -(x + y)/xy$. By substitution we find, $\det H(\mathcal{J}) = 0$. ■

Theorem .5 \mathcal{D} is a linear measure.

Proof The Hession of $\mathcal{D}[x, y]$ is evaluated: $\partial^2 \mathcal{D} / \partial x \partial x = 1/x$, $\partial^2 \mathcal{D} / \partial y \partial y = x/y^2$, and $\partial^2 \mathcal{D} / \partial x \partial y = \partial^2 \mathcal{D} / \partial y \partial x = -1/y$. By substitution we find, $\det H(\mathcal{D}) = 0$. ■

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